

- Please do not open the exam before you are instructed to do so.
- **Electronic devices are forbidden on your person**, including cell phones, tablets, headphones, and laptops. Turn your cell phone off and **leave all electronics at the front of the room**, or risk getting a zero on the exam.
- The exam is closed book, closed notes except your one-page cheat sheet. You are allowed one double-sided handwritten 8.5x11 inch cheatsheet.
- You have 1 hour and 50 minutes (unless you are in the DSP program and have an allowance of 150% or 200% time).
- Please write your initials at the top right of each page after this one (e.g., write “JD” if you are John Doe). Finish this by the end of your 1 hour and 50 minutes.
- Mark your answers on the exam itself in the space provided. Do **not** attach any extra sheets.
- For multiple answer questions, fill in the bubbles for **ALL correct choices**: there may be more than one correct choice, but there is always at least one correct choice. **NO partial credit** on multiple answer questions: the set of all correct answers must be checked.
- The last question (Question 8) is for CS289A students only. Students enrolled in CS189A will **not** receive any credit for answering this question.

First name	
Last name	
SID	
First and last name of student to your left	
First and last name of student to your right	

- CS 189A
- CS 289A

# 1 Multiple Choice (Single Answer)

- (1 point) Peanut wants to train a model to accurately classify different types of animals from images. After training and testing his model, he observes that the model has high training error and high test error. What can we most confidently say about the bias/variance characteristics of Peanut's model?
  - High bias.
  - Low bias.
  - High variance.
  - Low variance.
- (1 point) Consider a binary classification data set with 9000 positively labelled examples and 1000 negatively labelled examples. What is the area under the ROC curve (AUC-ROC) of a random classifier that classifies any example as positive with probability  $\pi$  and as negative with probability  $1 - \pi$ ? Here, the probability  $\pi$  is a hyperparameter.
  - Close to zero.
  - Close to 0.1.
  - Close to 0.5.
  - Close to 0.9.
  - Close to one.
- (1 point) Again, consider a binary classification data set with 9000 positively labelled examples and 1000 negatively labelled examples. What is the precision and the recall of a classifier that always classifies any example as positive?
  - The precision is 0.1, and the recall is 0.9.
  - The precision is 0.9, and the recall is 0.1.
  - The precision is 1.0, and the recall is 0.9.
  - The precision is 0.9, and the recall is 1.0.
  - The precision is 0.1, and the recall is 1.0.

4. (2 points) Assume we are given  $X \in \mathbb{R}^{n \times d}$  and  $y \in \mathbb{R}^n$  for  $n > d$ . The Ridge regression estimator with regularization coefficient  $\lambda$  estimates the weight vector to be

$$\hat{w} = \arg \min_w \left( \|y - Xw\|_2^2 + \lambda \|w\|_2^2 \right). \quad (1)$$

The Ridge regression estimator is equivalent to the ordinary least squares estimator on which of the following modified version of  $X$  and  $y$ ?

$I_d$  denotes the  $d \times d$  identity matrix.  $0_d$  denotes the all-zero  $d$ -dimensional vector, and  $1_d$  denotes the all-one  $d$ -dimensional vector.

A.

$$y' = \begin{bmatrix} y \\ 0_d \end{bmatrix}, X' = \begin{bmatrix} X \\ \sqrt{\lambda} I_d \end{bmatrix} \quad (2)$$

B.

$$y' = \begin{bmatrix} y \\ 1_d \end{bmatrix}, X' = \begin{bmatrix} X \\ \sqrt{\lambda} I_d \end{bmatrix} \quad (3)$$

C.

$$y' = \begin{bmatrix} y \\ 0_d \end{bmatrix}, X' = \begin{bmatrix} X \\ \lambda I_d \end{bmatrix} \quad (4)$$

D.

$$y' = \begin{bmatrix} y \\ 1_d \end{bmatrix}, X' = \begin{bmatrix} X \\ \lambda I_d \end{bmatrix} \quad (5)$$

## 2 Multiple Choice (Multiple Answer)

Fill in the bubbles for **ALL correct choices**: there may be more than one correct choice, but there is always at least one correct choice. **NO partial credit** on multiple answer questions: the set of all correct answers must be checked.

1. (2 points) Which of the following statements are **TRUE** regarding positive semi-definite and positive-definite matrices?
  - “Every entry of a matrix is non-negative” is a necessary but insufficient condition for a matrix to be positive definite.
  - The singular values of a positive semi-definite matrix are the same as its eigenvalues.
  - If a matrix  $A$  is positive semi-definite, then there exists a matrix  $B$  such that  $B^T B = A$ .
  - The covariance matrix of any distribution is positive semi-definite and invertible.
  - If the Hessian of a function is positive semi-definite, then the function is convex.
2. (2 points) Which of the following statements are **TRUE** regarding Lasso and Ridge regression? Let  $X \in \mathbb{R}^{n \times d}$  be the data matrix and  $y \in \mathbb{R}^n$  be the observed labels for the  $n$  examples. Let  $w \in \mathbb{R}^d$  be the weight parameters learned from Lasso or Ridge regression, and let  $\lambda$  be the regularization coefficient.
  - In the Bayesian MAP interpretation, Lasso regression can be interpreted as linear regression for which the coefficients have Poisson prior distributions.
  - In Lasso regression, as the regularization coefficient becomes very large ( $\lambda \rightarrow \infty$ ), the learned weights from Lasso regression will be close to zero ( $w \rightarrow 0$ ).
  - Lasso regression performs both feature selection and regularization.
  - There is no unique solution to Ridge regression whenever  $X$  is not full rank.
3. (2 points) If the model resulting from Ridge regression is currently overfitting, what are possible things to reduce overfitting? Select all that apply.
  - Collect new data to increase the training data size.
  - Repeat the current data twice to increase the training data size.
  - Increase the  $\ell_2$  regularization penalty in the loss function.
  - Add new features to the model.
  - Remove features from the model.
4. (2 points) Which of the following statements about gradient descent are **TRUE**?
  - After a gradient descent update step, the objective function value at the weight vector is always lower after the update than before.
  - There is always a unique steepest descent direction in gradient descent.
  - Gradient descent converges to a globally optimal solution for logistic regression under appropriate assumptions.

- Since ReLU is a convex function, a neural network that uses ReLU activations is also a convex function, and therefore gradient descent will converge to a globally optimal solution on neural networks with ReLU activations.
5. (2 points) Which of the following statements are **TRUE** about cross entropy?
- The value of cross-entropy loss is always non-negative.
  - Cross-entropy loss is only suitable for binary classification but not for multiclass classification.
  - For two discrete probability distributions  $P$  and  $Q$ , the KL divergence is symmetric, i.e.  $D_{KL}(P||Q) = D_{KL}(Q||P)$ .
  - Minimizing the cross-entropy is equivalent to maximizing the likelihood with respect to the model parameters.
6. (2 points) Which of the following statements are **TRUE** about cross validation? In this problem, the final test set is **not** involved in the cross validation process.
- During the  $k$ -fold cross validation process, precisely  $k$  models are trained on different subsets of the data.
  - During the  $k$ -fold cross validation process, precisely  $k - 1$  models are trained on different subsets of the data.
  - During the  $k$ -fold cross validation process, we need to draw  $k$  random seeds to shuffle the data precisely  $k$  times.
  - At the end of the  $k$ -fold cross validation process, we choose hyperparameters that minimize the highest validation loss among the different splits.
7. (2 points) Which of the following statements are **TRUE** about principle component analysis (PCA)? Given  $n$  data points of dimension  $d$ ,  $X \in \mathbb{R}^{n \times d}$ , assume that we are performing PCA to reduce the dimension of the data to  $k$  where  $k < d$ . Which of the following would result in a **DIFFERENT** PCA basis? Recall that the PCA basis is a set of unit vectors.
- Multiplying all columns in  $X$  by a factor of two before performing PCA.
  - Multiplying the first column in  $X$  by a factor of two before performing PCA.
  - Replacing the first column of  $X$  by the sum of all columns (including the first column).
  - Not subtracting the mean from  $X$  before performing PCA.
8. (2 points) Which of the following statements are **TRUE** about weight updates in neural networks?
- Typically, the weights in all hidden layers are initially all set to zero.
  - If using the mean squared error loss, the weight changes in the last layer are proportional to the difference between the model output and the true labels.
  - The weight changes in a particular hidden layer are proportional to the input to that weight layer.
  - Weight updates are computed in the forward pass of the network.

### 3 Fishing

1. (6 points) Rumble is fishing by a lake and catches  $n$  fish. She records the time she must wait in between catching each fish as  $x_i$  minutes, where  $x_i$  denotes the time she had to wait between catching fish  $(i - 1)$  and fish  $i$ . After catching  $n$  fish and collecting data points  $x_1, \dots, x_n$ , Rumble heads home for dinner. Rumble knows that the  $x_i$ 's are distributed according to a Poisson distribution,  $\text{Poisson}(\lambda)$ , for some unknown rate  $\lambda \in \mathbb{R}_+$ .

Recall that the probability mass function for a Poisson distribution with rate  $\lambda$  is

$$f_\lambda(x) = \frac{\lambda^x e^{-\lambda}}{x!}. \quad (6)$$

- (a) (4 points) With a prior  $\lambda \sim \text{Exp}(\xi)$  for some fixed  $\xi > 0$ , what is the maximum a posteriori (MAP) estimate for the rate  $\lambda$ ?

Recall that the exponential distribution  $\text{Exp}(\xi)$  has probability density function

$$f(\lambda) = \xi e^{-\xi\lambda} \text{ for } \lambda \geq 0.$$

- (b) (2 points) In Discussion 1, we have derived that the maximum likelihood estimation of  $\lambda$  is the sample mean

$$\hat{\lambda}_{MLE} = \frac{1}{n} \sum_{i=1}^n x_i. \quad (7)$$

How does the above MAP estimator compare to the MLE estimator in their bias and variance characteristics?

- A. The above MAP estimator has higher bias and lower variance.
- B. The above MAP estimator has lower bias and lower variance.
- C. The above MAP estimator has higher bias and higher variance.
- D. The above MAP estimator has lower bias and higher variance.

## 4 Watermelons

1. (6 points) Finn lives on a watermelon farm and wants to classify whether a watermelon is sweet (labelled as  $y = 1$ ) or not sweet (labelled as  $y = 0$ ). Finn observes a  $d$ -dimensional feature vector  $x \in \mathbb{R}^d$  associated with the appearance and the smell of each watermelon. Before observing a watermelon, Finn's general prior is that a watermelon is sweet with probability  $p(y = 1) = \pi_1$  and not sweet with probability  $p(y = 0) = 1 - \pi_1$ .

Finn is a watermelon expert and knows that the conditional probability distribution of the watermelon features  $p(x|y = k)$  for class  $k$  (where  $k = 0, 1$ ) is a  $d$ -dimensional Gaussian distribution  $N(\mu_k, \Sigma)$  with mean  $\mu_k \in \mathbb{R}^d$  and covariance  $\Sigma \in \mathbb{R}^{d \times d}$ . Note that the same covariance matrix  $\Sigma$  is shared between the two classes.

$$f(x|y = k) = \frac{1}{\sqrt{(2\pi)^d |\Sigma|}} \exp\left\{-\frac{1}{2}(x - \mu_k)^T \Sigma^{-1} (x - \mu_k)\right\} \quad (8)$$

Can you help Finn find out how likely a given watermelon is sweet? Write down a simplified expression of  $p(y = 1|x)$  as a function of  $x, \mu_0, \mu_1, \Sigma$ , and  $\pi_1$ . The expression should be in the form of  $s(w^T x + b)$  where  $s(z) = \frac{1}{1+e^{-z}}$  is the sigmoid function. Write down what  $w$  and  $b$  should be.

## 5 Road Bikes and Mountain Bikes

1. (8 points) In statistics, a mixture model is a probabilistic model for representing the presence of subpopulations within an overall population. For example, we might observe the prices of  $n$  different bikes without knowing the type of each bike (*e.g.*, road bike or mountain bike). We could model the bike prices as a mixture model with two different components, where one component corresponds to road bikes and the other corresponds to mountain bikes.

Let us assume that the price distribution is Gaussian for each type of bikes. In this case, the overall price distribution of all bikes is a mixture model of two univariate Gaussian distributions, where the likelihood is given by

$$p(x | \pi_0, \pi_1, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2) = p(k = 0)f(x | \mu_0, \sigma_0^2) + p(k = 1)f(x | \mu_1, \sigma_1^2), \quad (9)$$

where  $p(k = 0) = \pi_0$  is the probability that a given bike is a road bike,  $p(k = 1) = \pi_1 = 1 - \pi_0$  is the probability that a given bike is a mountain bike, and  $f(x | \mu_k, \sigma_k^2)$  is the Gaussian probability density function with mean  $\mu_k$  and variance  $\sigma_k^2$ .

We observed  $n$  i.i.d. samples of bike prices  $x_1, \dots, x_n$  from this mixture of Gaussian distributions.

- (a) (2 points) What is the log probability of the bike prices  $\{x_i\}_{i=1}^n$ ? Write down an expression for  $\log p(x_1, \dots, x_n | \pi_0, \pi_1, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2)$ .



- (b) (2 points) Unfortunately, there is no closed-form expression for the maximum likelihood estimator for the likelihood function above. In one approach to estimate the parameters for a Gaussian mixture model, we introduce unobserved random variables  $c_i$  associated with the component of each  $x_i$ . The unobserved binary variable  $c_i \in \{0, 1\}$  indicates which of the two Gaussian distributions  $x_i$  is from (e.g., bike type).

$$P(c_i = 0) = \pi_0 \text{ and } P(c_i = 1) = \pi_1. \quad (10)$$

Conditioned on the component  $c_i$  (e.g., bike type), the observations  $x_i \mid c_i$  (e.g., bike prices) follows a Gaussian distribution  $N(x_i \mid \mu_k, \sigma_k^2)$ . For  $k = 0, 1$ ,

$$p(x_i \mid c_i = k) = \frac{1}{\sigma_k \sqrt{2\pi}} \exp\left\{-\frac{(x_i - \mu_k)^2}{2\sigma_k^2}\right\}. \quad (11)$$

Suppose that we already know the unobserved variables  $c_1, \dots, c_n$  in addition to  $x_1, \dots, x_n$ . The joint density of the samples and the unobserved components  $\{(x_i, c_i)\}_{i=1}^n$  is known as the *complete likelihood*.

Find an expression for  $\log p((x_1, c_1), \dots, (x_n, c_n) \mid \pi_0, \pi_1, \mu_0, \sigma_0^2, \mu_1, \sigma_1^2)$ . In your answer, you may use notations such as  $\mu_{c_i}$ .

- (c) (4 points) Still under the assumption that the unobserved component variables  $c_1, \dots, c_n$  are known, what is the maximum likelihood estimation (MLE) of  $\mu_0, \sigma_0^2, \mu_1$ , and  $\sigma_1^2$ ? In your answer, write down expressions for  $\hat{\mu}_k$  and  $\hat{\sigma}_k^2$  in a way that holds for both  $k = 0, 1$ . Let  $S_k := \{i : c_i = k \text{ for } i = 1, \dots, n\}$  denote the set of all  $i$ 's such that  $c_i = k$ . For example,  $S_0$  is the set of indices for all examples from the 0-th component. This is the maximization (M) step of the expectation-maximization (EM) algorithm for estimating the parameters of a Gaussian mixture model.

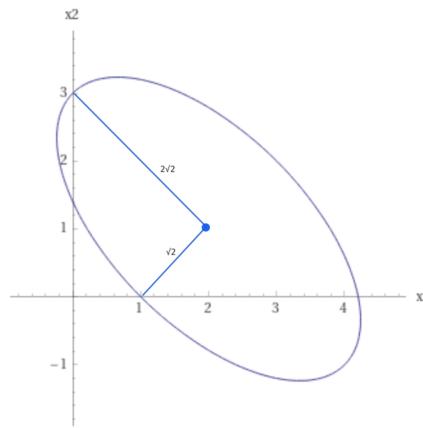
## 6 Level Sets

1. (7 points) Esme generated a 2D multivariate Gaussian distribution but lost the parameters used for generating the multivariate Gaussian distribution. The only piece of paper left has a plot of the level set of the probability density function  $f$  at value

$$f(x_1, x_2) = \frac{1}{e}(\text{max Gaussian density}).$$

Here, “max Gaussian density” refers to the maximum value of the Gaussian probability density function. A picture of the level set is shown below. For a function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  and a constant  $c \in \mathbb{R}$ , the level set is defined as  $\{x \in \mathbb{R}^2 : f(x) = c\}$ .

The level set on Esme’s paper is an ellipse centered at  $(2, 1)$  with axes along the unit vectors  $(1, -1)$  and  $(1, 1)$ . The two axes correspondingly have radii  $r_1 = 2\sqrt{2}$  and  $r_2 = \sqrt{2}$  in the ellipse.

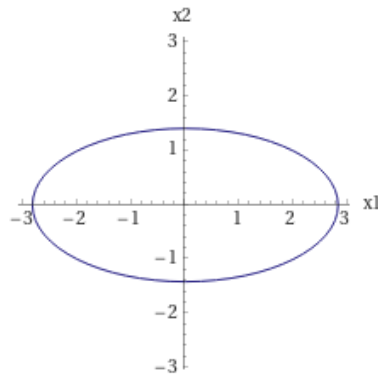


- (a) (1 point) What is the mean of the multivariate Gaussian distribution?
- (b) (1 point) Suppose we have a 2D Gaussian distribution with zero mean and diagonal covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}.$$

What is the maximum density in this distribution?

- (c) (2 points) Consider a rotated version of the ellipse that is aligned with the two axes and centered at zero (as illustrated below). As before, the radii of the ellipse are  $2\sqrt{2}$  and  $\sqrt{2}$ . What is the multivariate Gaussian distribution that has the rotated ellipse as its level set at  $1/e$  of the maximum density?



Recall that the equation of an ellipse centered at zero and aligned with the  $x_1$ - and  $x_2$ - axis radii  $a$  and  $b$  is

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1. \quad (12)$$

- (d) (3 points) What is the covariance matrix of the multivariate Gaussian distribution for the level set described in the beginning of the question? It is sufficient to write the covariance matrix in the form of  $\Sigma = QDQ^T$ .

*Hint: The determinant of the covariance matrix  $\Sigma$  is the product of its eigenvalues. Think also about the spectral decomposition of  $\Sigma$ .*

## 7 Activation Functions

1. (6 points) While Rectified Linear Unit (ReLU) is often the default activation function used across the deep learning community, other activation functions have also been proposed to replace ReLU. Consider the following activation function (known as *Swish* or SiLU):

$$f(x) = x \cdot \sigma(\beta x), \quad (13)$$

where  $\sigma(z) = \frac{1}{1+e^{-z}}$  is the sigmoid function and  $\beta$  is a fixed hyperparameter.

Like ReLU, this activation function is unbounded above and bounded below. Unlike ReLU, this activation function is smooth and non-monotonic.

- (a) (1 point) What is the derivative  $f'(x)$  of this new activation function (for a fixed  $\beta$ )?
- (b) (2 points) What does this activation function look like for  $\beta = 1$  and for  $\beta \rightarrow \infty$ ? Draw qualitative pictures for both values of  $\beta$ .

Consider a binary classification data set of 20000 examples. Among the 20000 examples, 5000 of them have the positive class label, and the remaining 15000 have negative class labels. Suppose we have a fully-connected neural network with one hidden layer and with sigmoid activation after the output layer. When training this neural network for one epoch using stochastic gradient descent (SGD), we forget to shuffle the examples and loop through all the negative examples first before the positive examples.

- (c) (1 point) After the model is trained on the 15000 examples with negative class labels, which of the following is likely true?
- A. The model output **before** the sigmoid activation would always be close to zero.
  - B. The model output **before** the sigmoid activation would always be very negative.
  - C. The model output **before** the sigmoid activation would always be very positive.
  - D. The model output **before** the sigmoid activation would sometimes be positive and sometimes be negative.
- (d) (1 point) Suppose we use ReLU as the activation function in the hidden layer. In the above scenario, after training on only negatively-labelled examples, the activation input to the ReLU function might end up being always negative. What would then happen when the model is trained on the positive-labelled examples afterwards? Describe qualitatively in 1-2 sentences.
- (e) (1 point) Answer the above question with  $f(x) = x \cdot \sigma(\beta x)$  as the activation function with  $\beta = 1$ .

## 8 Projected Gradient Descent for Lasso Regression (CS 289A Only)

1. (6 points) Sometimes, we may wish to add constraints to an optimization problem. For example, we might want certain variables to be non-negative. Or, we might need to satisfy budget constraints and resource constraints.

In this problem, we will consider Lasso regression. Let  $X \in \mathbb{R}^{n \times d}$  be the features and  $y \in \mathbb{R}^n$  be the labels of a regression problem. Given a regularization coefficient  $\lambda$ , the objective of Lasso regression is to solve

$$\min_{w \in \mathbb{R}^d} \frac{1}{n} \|y - Xw\|_2^2 + \lambda \|w\|_1. \quad (14)$$

Unlike Ridge regression, the loss function for Lasso regression is not differentiable. One way to tackle the optimization problem for Lasso is to rewrite the above objective into the following smooth, constrained optimization problem (by separating  $w$  into  $w = w^+ - w^-$  for two non-negative vectors  $w^+$  and  $w^-$  representing the positive and negative parts of  $w$ ):

$$\begin{aligned} \min_{w^+, w^- \in \mathbb{R}^d} \frac{1}{n} \|y - X(w^+ - w^-)\|_2^2 + \lambda \mathbb{1}_d^T w^+ + \lambda \mathbb{1}_d^T w^- \\ \text{subject to: } w_i^+, w_i^- \geq 0 \text{ for } i = 1, \dots, d. \end{aligned} \quad (15)$$

$\mathbb{1}_d$  denotes the all-one  $d$ -dimensional vector and therefore  $\mathbb{1}_d^T w^+ = \sum_{i=1}^d w_i^+$ .

In general, a constrained minimization problem for a given constraint set  $C \subset \mathbb{R}^d$  asks for the best solution  $\min_{w \in C} f(w)$  inside the set  $C$ . In the above constrained formulation of Lasso regression, if we concatenate  $w_i^+$  and  $w_i^-$  together as a new variable  $u = [w_i^+ \ w_i^-] \in \mathbb{R}^{2d}$ , the constraint set is the positive orthant in  $\mathbb{R}^{2d}$

$$C = \{u \in \mathbb{R}^{2d} : u_i \geq 0 \text{ for } i = 1, \dots, 2d\}. \quad (16)$$

We have seen that gradient descent is a standard way to solve unconstrained optimization problem. Here, we will see that we can modify gradient descent into projected gradient descent to solve constrained optimization problems.

- (a) (2 points) Ignoring the constraint for a moment, what is one step of the gradient descent update for  $w^+$  and  $w^-$  in the following unconstrained regression problem?

$$\min_{w^+, w^- \in \mathbb{R}^d} \frac{1}{n} \|y - Xw^+ + Xw^-\|_2^2 + \lambda \mathbb{1}_d^T w^+ + \lambda \mathbb{1}_d^T w^- \quad (17)$$

Assume that the step size is  $\eta$ .



(b) (2 points) Now back to the constrained optimization problem. When  $C$  is a convex set, the projection operator  $P_C : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$  onto a convex set  $C$  is defined for  $a \in \mathbb{R}^{2d}$  as

$$P_C(a) := \arg \min_{b \in C} \|a - b\|_2^2. \quad (18)$$

The projection onto a convex set is uniquely defined. Give an expression (without proof) for the projection  $P_C$  onto the positive orthant

$$C = \{a \in \mathbb{R}^{2d} : a_i \geq 0 \text{ for } i = 1, \dots, 2d\}. \quad (19)$$

- (c) (2 points) Projected gradient descent introduces an additional projection in every gradient descent update, which aims to minimize loss while guaranteeing that the weight stays within the constraint set  $C$ . For the constraint set  $C \subset \mathbb{R}^{2d}$ , starting from a initial point  $u_0 \in C$ , with stepsize  $\eta > 0$ , projected gradient descent iterates the following equation until a stopping condition is met:

$$u_{k+1} = P_C(u_k - \eta \nabla f(u_k)). \quad (20)$$

For the reformulated Lasso optimization problem in Equation 15, Write down an expression for the projected gradient descent update for  $w_{k+1}^+$ ,  $w_{k+1}^-$  as a function of  $w_k^+$  and  $w_k^-$ ,  $X$ ,  $Y$ ,  $\lambda$ , and  $\eta$ .